Problem Solving 3

Lecture 14 Apr 11, 2021

- Q1. Let A be a set of natural numbers so that the sum of every 3 numbers in A is a prime.

What is the maximum possible size of such a set?

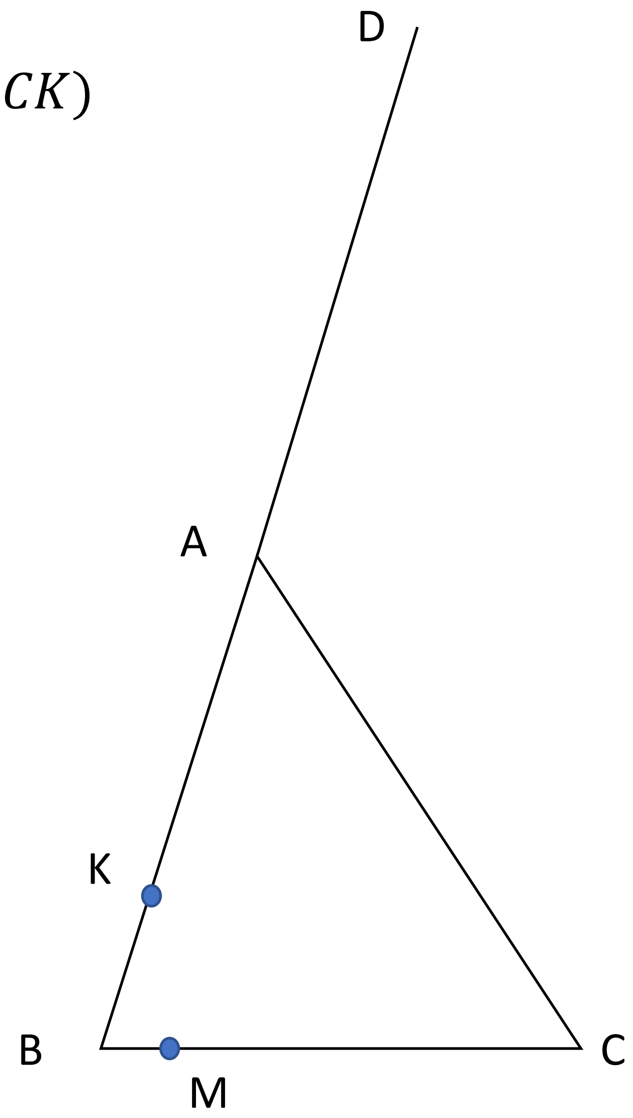
- Q2. Find all the solutions of the equation

$$(x+1)^2 = x^y + 1$$

- Q3. Let ABC be a triangle with $\angle A = 45^o$. Suppose D is a point on the continuation of the line segment BA such $BD = BA + AC$. Also, suppose K and M are points on AB and BC such that

$$area(BDM) = area(BCK)$$

Find the angle $\angle BKM$



- Q4. Find all the solutions of the equation

$$n! + 3 = 3^{n-1}$$

- Q5. We have 5 white and 10 black coins. In how many ways we can arrange them in a line so that after each white coin there is at least one black coin.

- Q6. For any selection of initial real numbers $a_1, a_2, \ldots, a_n$ we play the following game: in each turn, we choose two of them, say x and y, and replace them with $x + y + xy$.

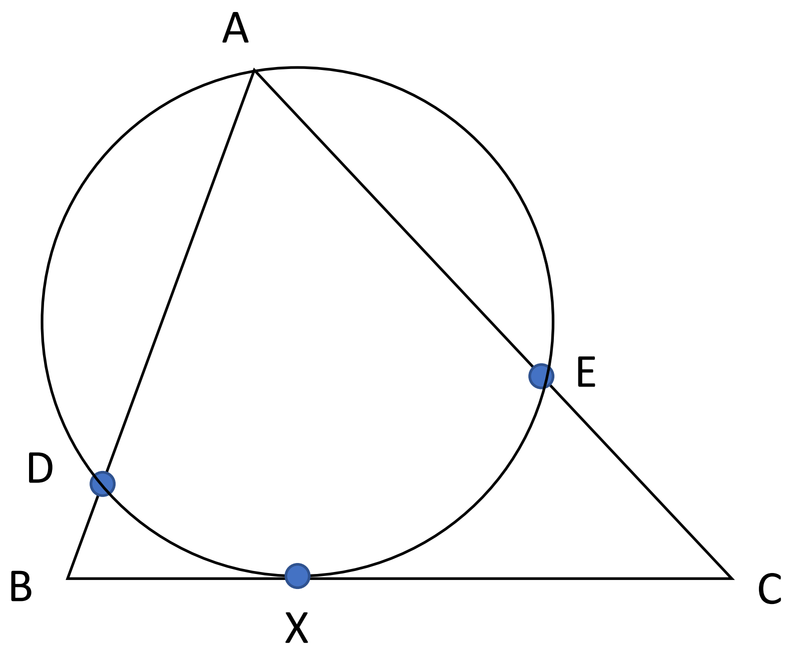
Therefore, after $n - 1$ steps only one number, say A, is left.

Which of the following conclusions about A is correct:

- ❑ There is a choice of initial numbers $a_1, a_2, \ldots, a_n$ for which there are $n!$ possible outcomes for A
- ❑ No matter how we play, A will always be the same
- ❑ If one of the numbers is even, A will certainly be even
- ❑ There are finitely many choices of $a_1, a_2, \ldots, a_n$ for which A will be 1
- ❑ The statements above are all wrong!

- Q7. Suppose the triangle ABC has acute angles. Let D and E be points on AB and AC such that the circumcircle of the triangles ADE is tangent to BC at the point X. If we choose D and E so that the distance DE is minimum, then which of the following statements is correct:

- ❑ X is the midpoint of BC
- ❑ AX is perpendicular to BC
- ❑ AX is the angle bisector of A
- ❑ None



- Q8.
- We have 2021 lamps $L_1, \dots, L_{2021}$ that are all off at the beginning.
- We also have 2021 switches $P_1, \dots, P_{2021}$.
- Pressing the switch P_k will turn On/Off all the lamps L_n where n is a multiple of k. (For example, L_2 will turn On/Off all even numbered lamps).

- If we press all of $P_1, \dots, P_{2021}$ once in order, at the end, how many lamps will be ON?